Asymptotic behavior of Aldous' gossip process

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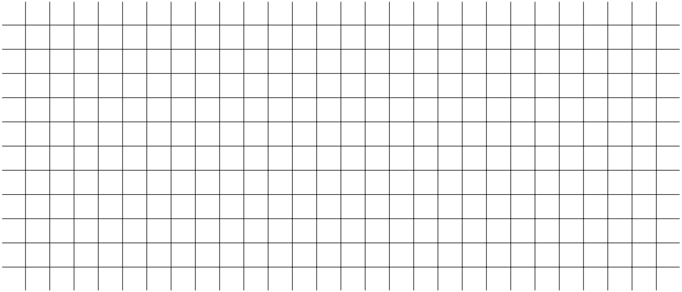
Joint work with Rick Durrett

Conference on Limit Theorems in Probability, IISC

First passage percolation model on $\mathbb{Z}^d$



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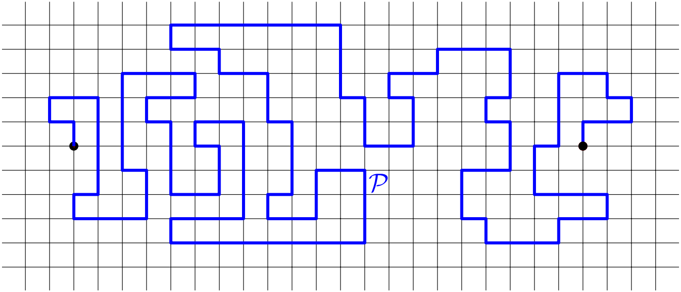

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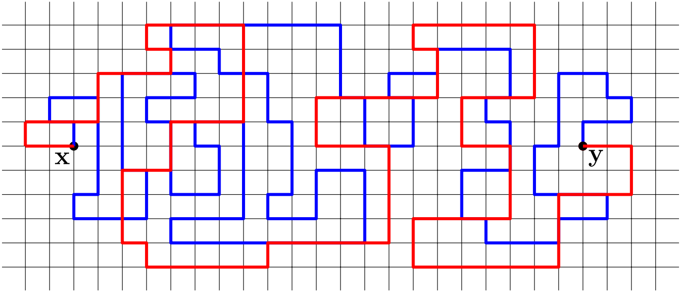

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- For a path $\mathcal{P}$, the passage time for $\mathcal{P}$ is defined as the sum of weights over all the edges in $\mathcal{P}$.
- For $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, the first-passage time $a(\mathbf{x}, \mathbf{y})$ is defined as the minimum passage time over all paths from $\mathbf{x}$ to $\mathbf{y}$.

Mean behavior

- The model was introduced by Hammersley and Welsh in 1965, where they proved that for all $\mathbf{x} \in \mathbb{Z}^d$

$$\nu(\mathbf{x}) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(a(\mathbf{0}, n\mathbf{x}))$$

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- The shape theorem by Cox and Durrett('81) says that

$$\frac{1}{t}\{\mathbf{x} \in \mathbb{Z}^d : a(\mathbf{0}, \mathbf{x}) \le t\} \oplus \left[-\frac{1}{2}, \frac{1}{2}\right]^d \xrightarrow{t \to \infty} B$$

  where $B = \{\mathbf{x} : \nu(\mathbf{x}) \le 1\}$ is a convex subset in $\mathbb{R}^d$.

Shape Theorem

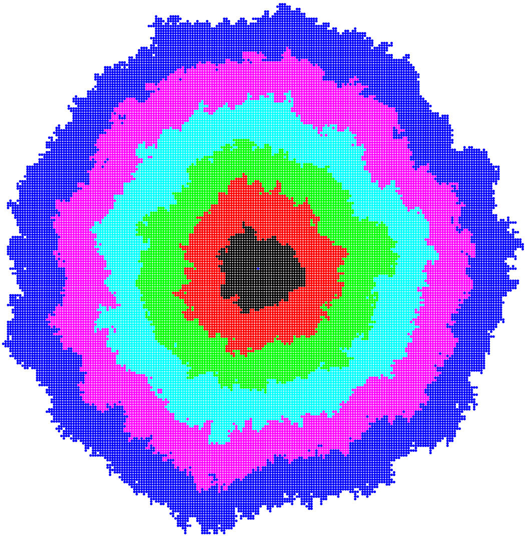

Figure: Limiting shape for $\{\mathbf{x} \in \mathbb{Z}^d : a(\mathbf{0}, \mathbf{x}) \leq t\}$.

First passage percolation on a torus

- Space is $\Lambda(N) = (\mathbb{Z} \mod N)^2$.
- Suppose one agent is present at each vertex of $\Lambda(N)$.
- At time 0 the center receives an information.
- Each neighbor of the center gets the information independently at a constant rate λ.
- In general, whenever a vertex is informed, each of its uninformed neighbor gets the information independently at rate λ.
- ξ_t is the set of vertices informed by time t. $\xi_0 = \{(0,0)\}$.

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Questions:

- How does ξ_t grow?
- When $\xi_t = \Lambda(N)$?

If T_N is the time when every agent on the torus is informed, then T_N/N converges to a number.

Short-long FPP on $\Lambda(N)$ (Aldous '07)

- State of the process is $\xi_t \subset \Lambda(N)$, the set of informed vertices at time t. $\xi_0 = \{(0,0)\}$.
- Information spreads from vertex i to j at rate ν_{ij}, where

$$\nu_{ij} = \begin{cases} 1/4 & \text{if } j \text{ is a (nearest) neighbor of } i \\ \lambda_N/(N^2-5) & \text{if not.} \end{cases}$$

- If a vertex gets the information from a non-neighbor, we call it a new 'center' for information spreading.
- So each informed vertex tries to spread the information locally at rate 1 and give birth to new centers at rate λ_N.

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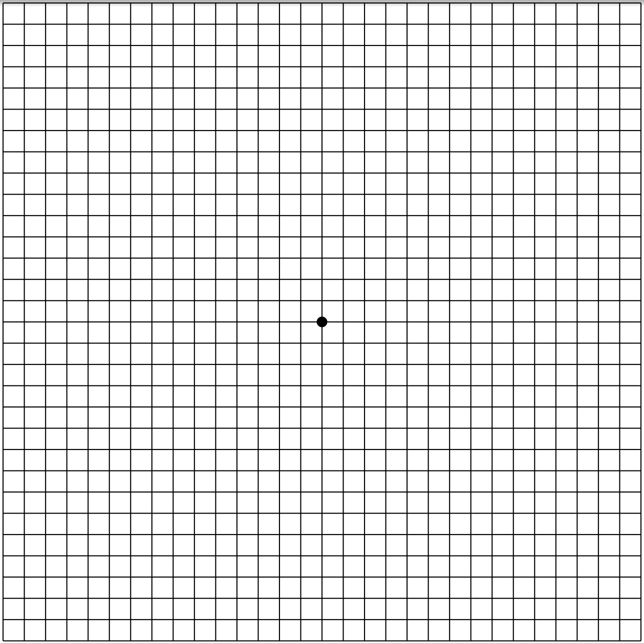
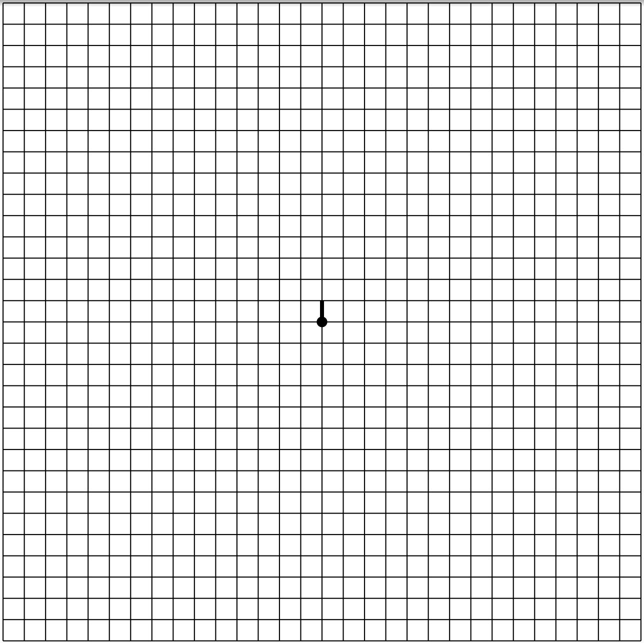
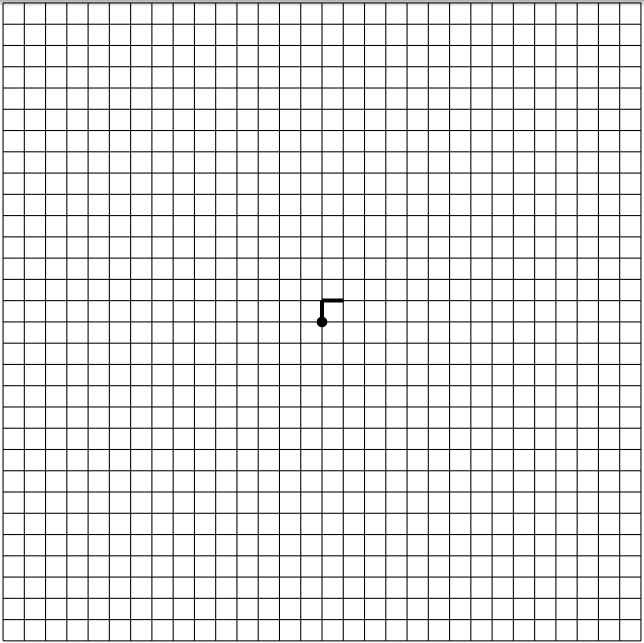
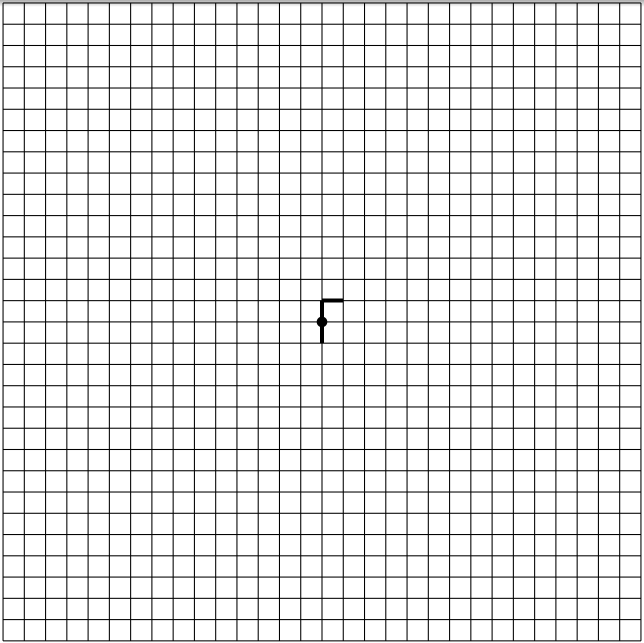
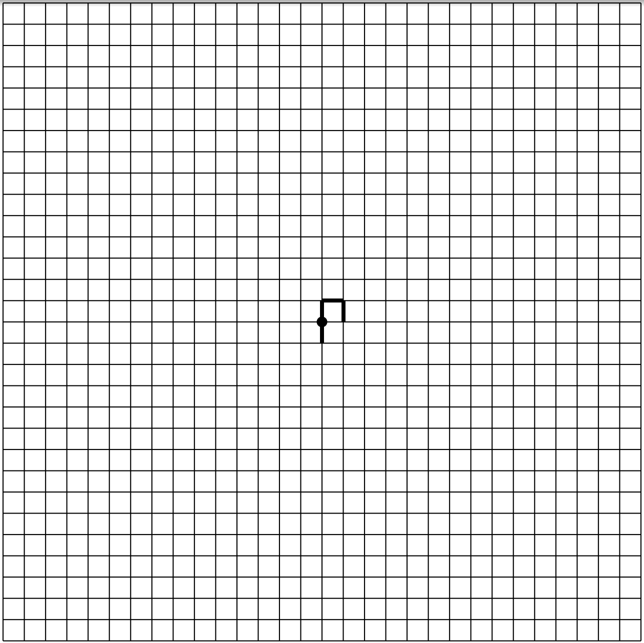
Question: How does ξ_t grow?
How does T_N (cover time of the torus) scale?

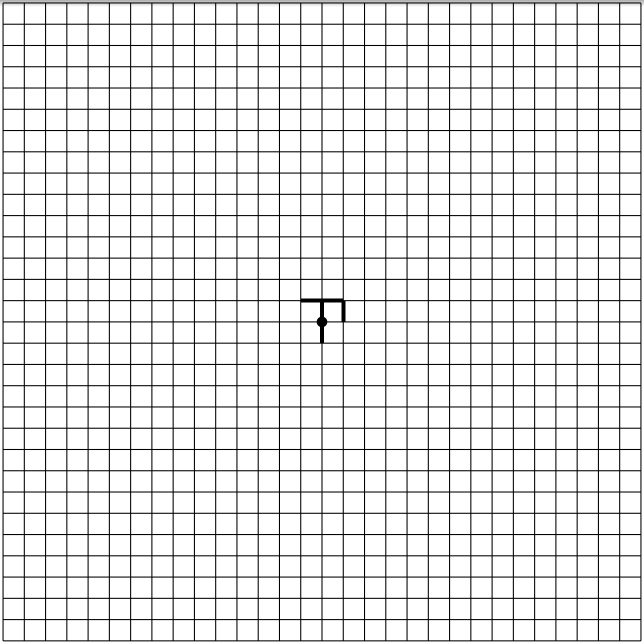


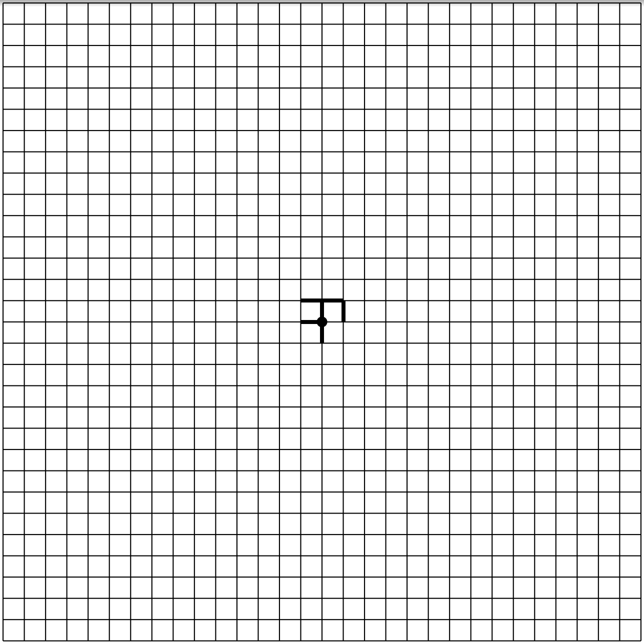


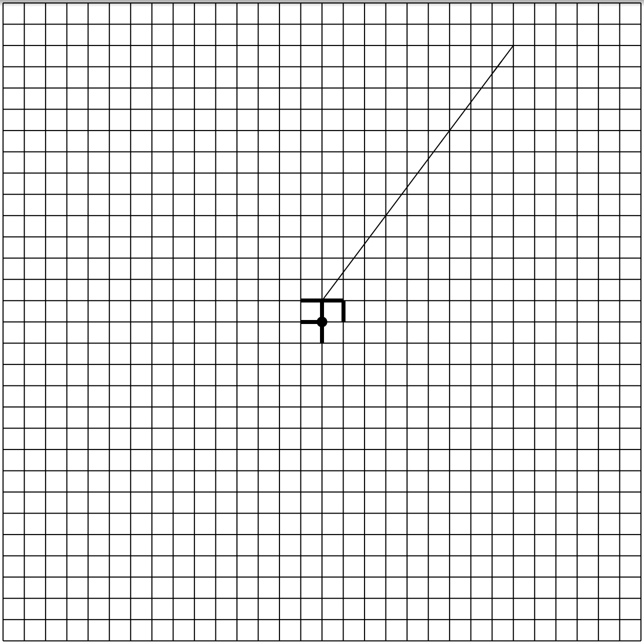


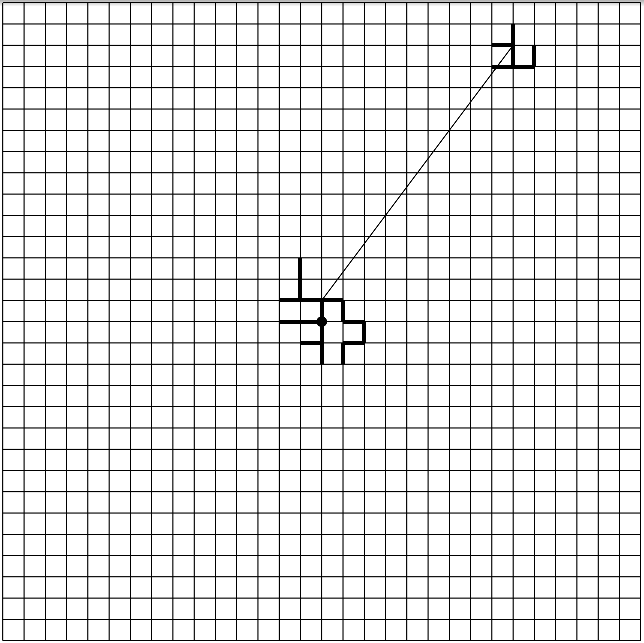


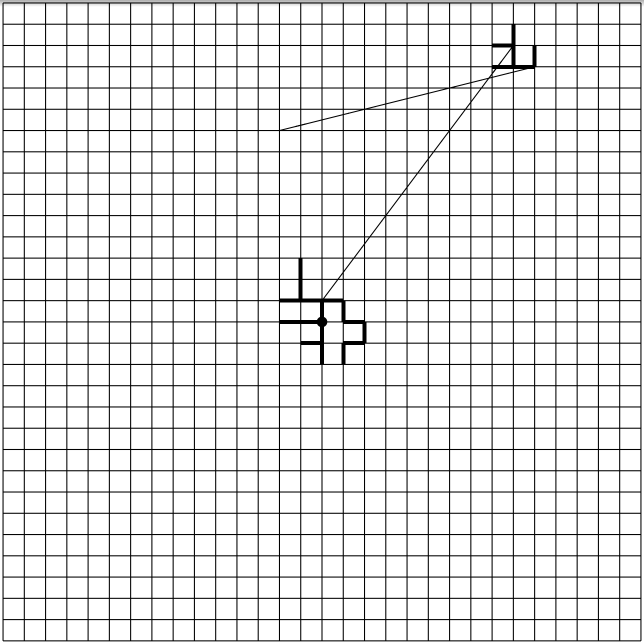


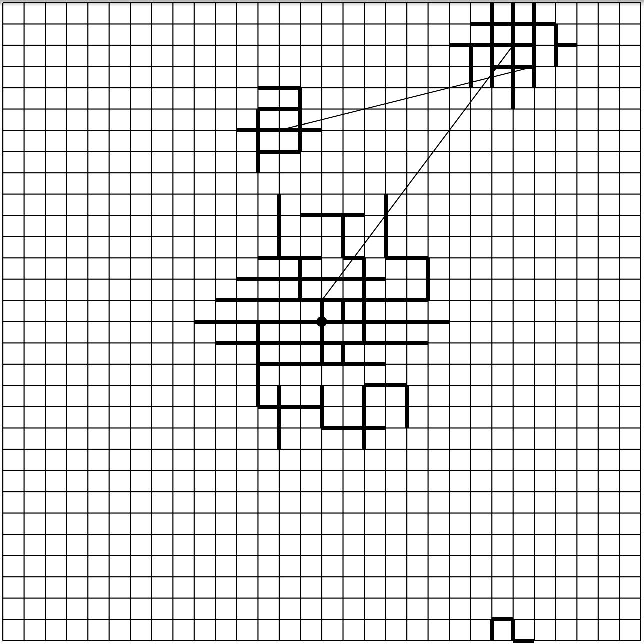


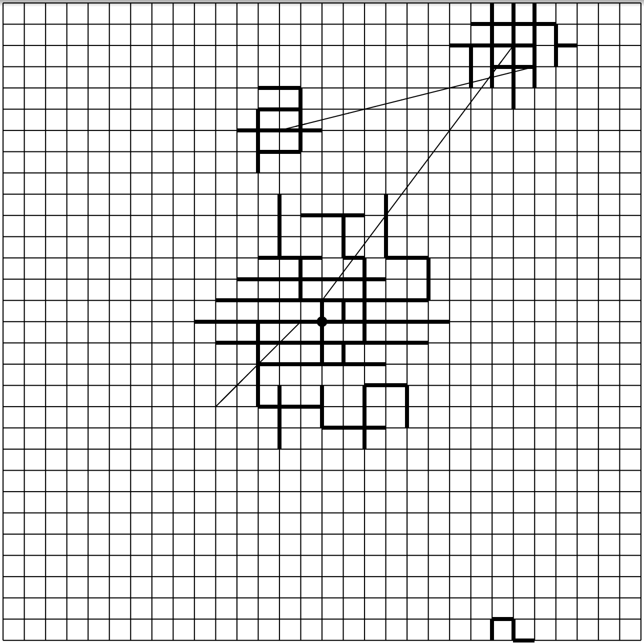


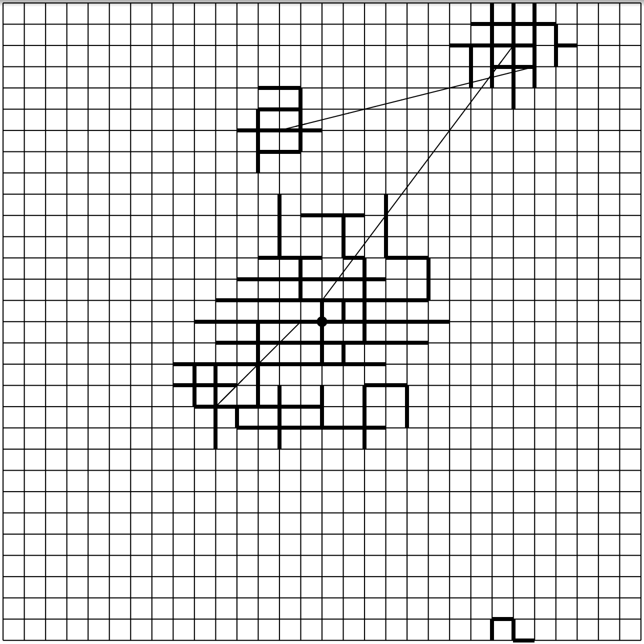












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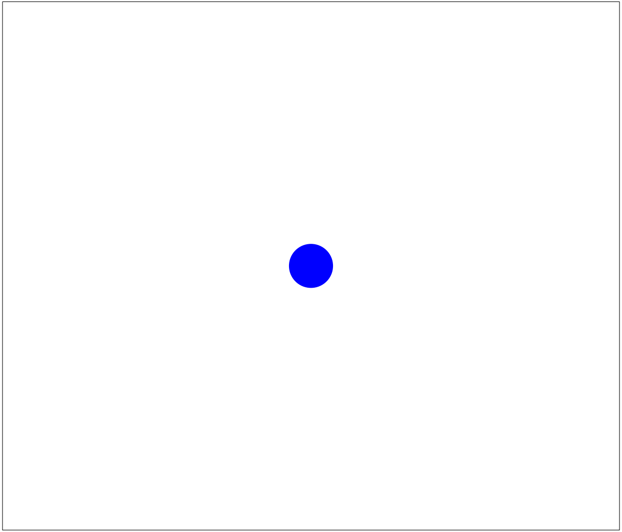
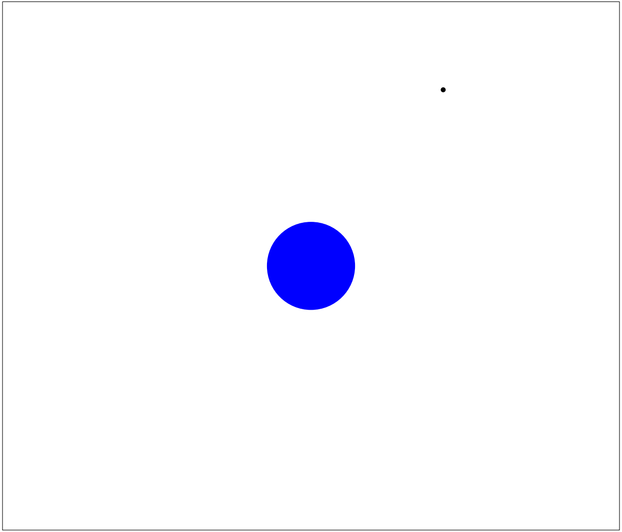
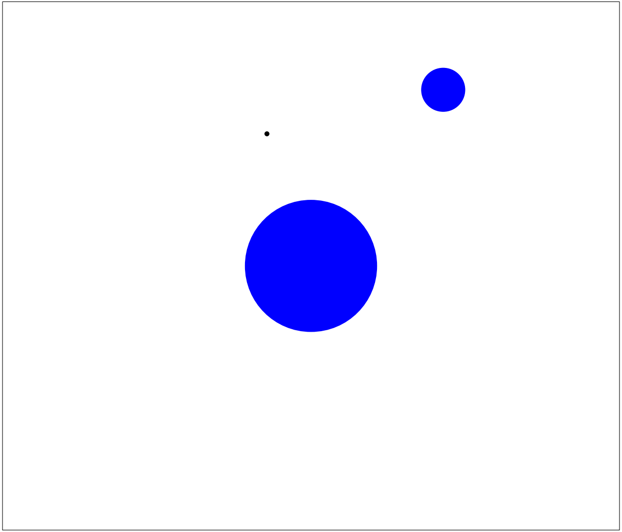
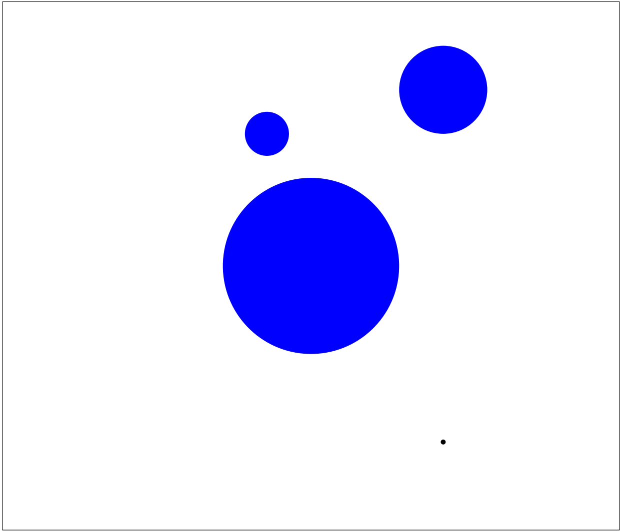
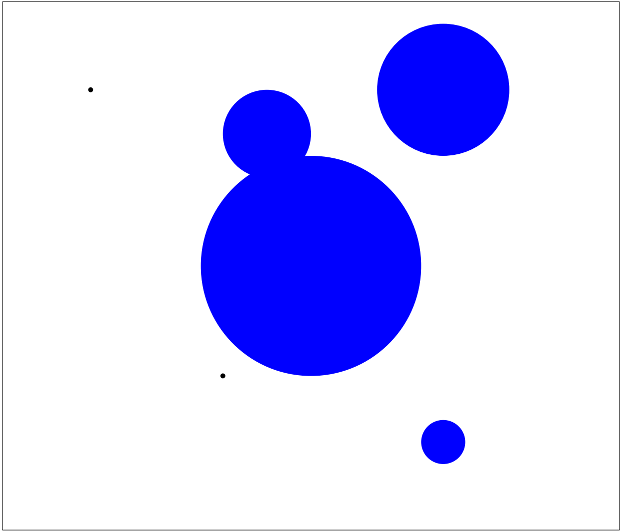
- we remove randomness from nearest neighbor growth
- we formulate on the (real) torus $\Gamma(N) = (\mathbb{R} \mod N)^2$.
- The state of our process at time t is $\mathcal{C}_t \subset \Gamma(N)$, the subset informed by time t.
- $\mathcal{C}_t$ starts with one *center* chosen uniformly from $\Gamma(N)$ at time 0.
- Each *center* is the center of growing disks, whose radius $r(\cdot)$ grows deterministically and linearly. We take $r(s) = s/\sqrt{2\pi}$.
- At time t, birth rate of new *centers* is $\lambda_N |\mathcal{C}_t| = \lambda_N C_t$.
- The location of each new *center* is chosen uniformly from the torus.
- If the new *center* lands at $x \in \mathcal{C}_t$, it has no effect. But we count all *centers* in $\tilde{X}_t$.

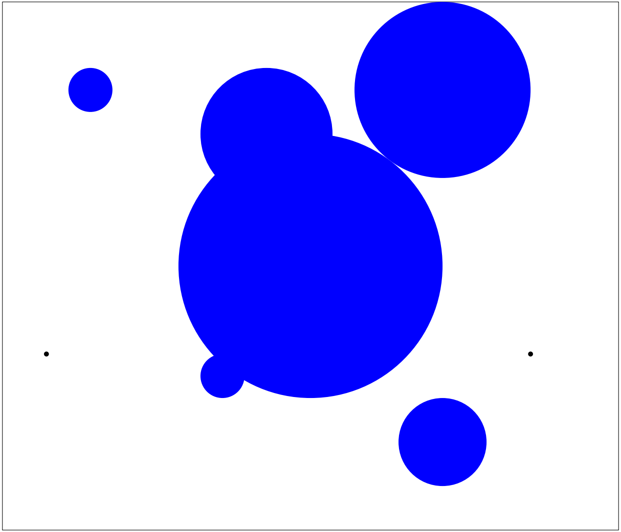


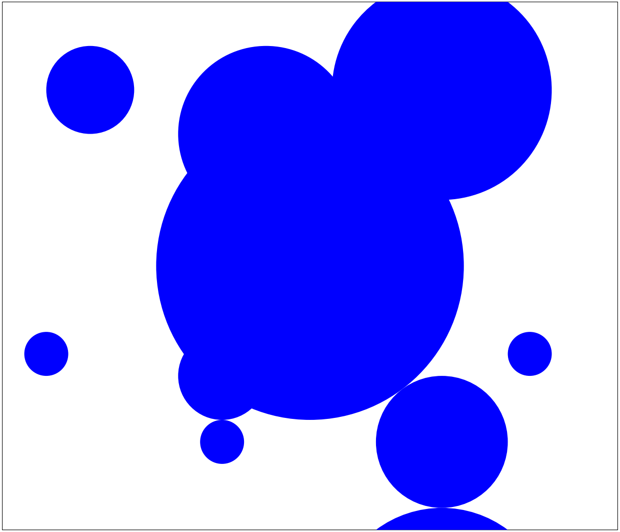


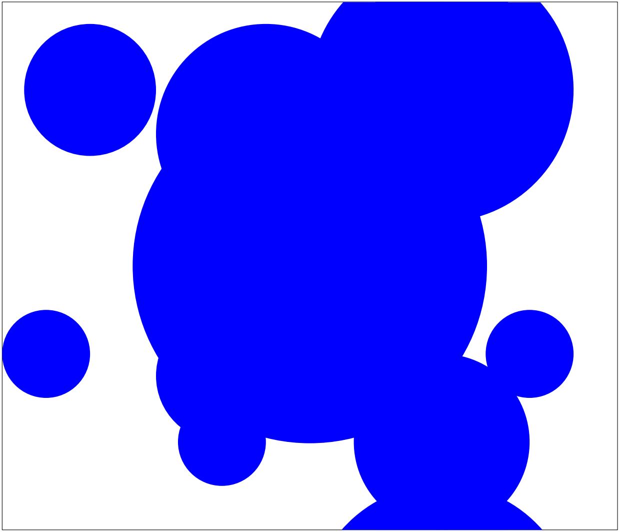












Phase transition

Consider $\lambda_N = N^{-\alpha}$.

- Case 1: $\alpha > 3$.
  - If the diameter of $\mathcal{C}_t$ grows linearly, then $\int_0^{c_0 N} C_t \, dt = O(N^3)$.
  - So w.h.p. no new *center* will be born before the initial disk covers the entire torus, and
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- Case 2: $\alpha = 3$.
  - with probabilities bounded away from 0, (i) no new *center* will be born and $T_N \approx \sqrt{\pi} N$, and (ii) there will be O(1) many landing close enough to $(N/2, N/2)$ to make $T_N \leq (1-\delta)\sqrt{\pi} N$.
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  - T_N/N converges weekly to a random variable with support $[0, \sqrt{\pi}]$ and an atom at $\sqrt{\pi}$.
- Case 3: $\alpha < 3$.
  - Many new *centers* will be born.
  - The cover time is significantly accelerated.

Asymptotic behavior in '$\alpha < 3$' case

Theorem (C. and Durrett; AoAP 2011)

- *For $\alpha < 3$, the cover time T_N satisfies*

$$\frac{T_N}{N^{\alpha/3} \log N} \xrightarrow{P} 2 - 2\alpha/3.$$

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- *More precisely, for* $\psi(t) = N^{\alpha/3}[(2 - 2\alpha/3)\log N - \log M] + N^{\alpha/3}t$ *and* $\delta > 0$,

$$\lim_{N\to\infty} P\left(\sup_{s\le t}|N^{-2}C_{\psi(s)} - h(s)| \le \delta\right) = 1.$$

Branching balloon process $\mathcal{A}_t$

- Overlaps among the disks in $\mathcal{C}_t$ make it difficult to study.
- We begin by studying much simpler balloon branching process $\mathcal{A}_t$.

In the process $\mathcal{A}_t$,

- we do not ignore any *center* (unlike in $\mathcal{C}_t$),
- X_t denotes the number of *centers* at time t,
- $A_t = \int_0^t (t-s)^2/2 \, dX_s =$ total area of all disks born by time t,
- new *centers* are born at rate $N^{-\alpha} A_t$ at uniformly chosen locations.

We couple $\mathcal{C}_t$ and $\mathcal{A}_t$ so that

- they start from the same point, and
- $\mathcal{C}_t \subset \mathcal{A}_t, C_t \leq A_t, \tilde{X}_t \leq X_t \; \forall t \geq 0$. (Recall $\tilde{X}_t = \#$ *centers* in $\mathcal{C}_t$)

Properties of $\mathcal{A}_t$

Let $\lambda = N^{-\alpha}$.

- Let $L_t := \int_0^t X_s \, ds$ be the length process. Then $A_t = \int_0^t (t-s)^2/2 \, dX_s = \int_0^t L_s \, ds$.
- Using i.i.d. behavior of all the centers,

$$X_t = 1 + \sum_{i : s_i \in \Pi_t} X^i_{t - s_i},$$

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- A little Poisson process computation shows that

$$EX_t = 1 + \int_0^t EX_{t-s}\, \lambda \frac{s^2}{2}\, ds, \text{ as area of initial disk at time } s \text{ is } \frac{s^2}{2}.$$

- Solving the renewal equation

$$EX_t = \sum_{k=0}^{\infty} \frac{\lambda^k t^{3k}}{(3k)!}.$$

Properties of $\mathcal{A}_t$ (continued)

- Solving the ODE $v''' = \lambda v$, (ω, ω^2 are complex cube roots of 1)

$$EX_t = \frac{1}{3}\left[\exp(\lambda^{1/3}t) + \exp(\lambda^{1/3}\omega t) + \exp(\lambda^{1/3}\omega^2 t)\right], \text{ and so}$$

$$EA_t = \frac{\lambda^{-2/3}}{3}\left[\exp(\lambda^{1/3}t) + \omega\exp(\lambda^{1/3}\omega t) + \omega^2\exp(\lambda^{1/3}\omega^2 t)\right],$$

- (X_t, L_t, A_t) is a Markov process.
- If $\mathcal{F}_s = \sigma\{X_r, L_r, A_r : r \leq s\}$, then

$$\frac{d}{dt}E\left[\begin{array}{c|c} X_t & \\ L_t & \mathcal{F}_s \\ A_t & \end{array}\right]\Bigg|_{t=s} = Q\begin{bmatrix} X_s \\ L_s \\ A_s \end{bmatrix}, \text{ where } Q = \begin{pmatrix} 0 & 0 & \lambda \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

- The left eigenvalues of Q are $\eta = \lambda^{1/3}, \lambda^{1/3}\omega, \lambda^{1/3}\omega^2$ with eigenvector $(1, \eta, \eta^2)$.
- From Dynkin;s formula, $e^{-\eta t}(X_t + \eta L_t + \eta^2 A_t)$ is a (complex) martingale.

The random variable M

Theorem

$M_t := \exp(-\lambda^{1/3}t)(X_t + \lambda^{1/3}L_t + \lambda^{2/3}A_t)$ is a positive L^2-martingale, and so

1. *$M_t \to M$ a.s. and in L^2,*
2. *M does not depend on λ and*
3. *$P(M > 0) = 1$,*
4. *$X_t/EX_t, L_t/EL_t, A_t/EA_t \to M$ a.s. and in L^2.*

Hitting time

- $\tau(\epsilon) = \inf\{t : C_t \geq \epsilon N^2\}$. We compare it with
$$\sigma(\epsilon) := \inf\{t : A_t \geq \epsilon N^2\}.$$
- $EA_t \sim a(t) = (1/3)N^{2\alpha/3}\exp(N^{-\alpha/3}t)$, and let
$S(\epsilon) := N^{\alpha/3}[(2-2\alpha/3)\log N + \log(3\epsilon)]$ so that $a(S(\epsilon)) = \epsilon N^2$.
- Using the L^2 convergence we have nice estimates for $P(\sup_{t\geq u}|A_t/a(t) - M| > \gamma)$ which in turn gives:

Lemma

If $0 < \epsilon < 1$, then as $N \to \infty$
$$N^{-\alpha/3}(\sigma(\epsilon) - S(\epsilon)) \xrightarrow{P} -\log(M).$$
The coupling between $\mathcal{C}_t$ and $\mathcal{A}_t$ implies $\tau(\epsilon) \geq \sigma(\epsilon)$.

Cone argument for upper bound for $\tau(\epsilon)$

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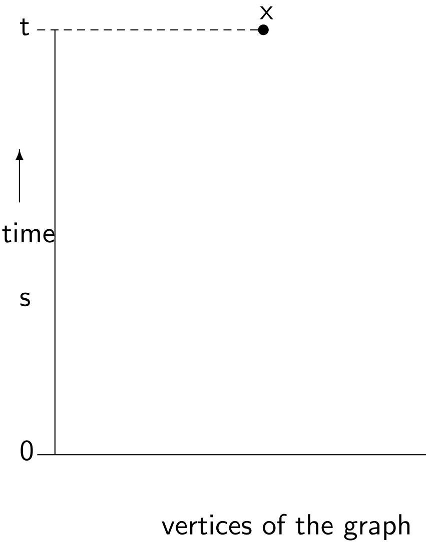
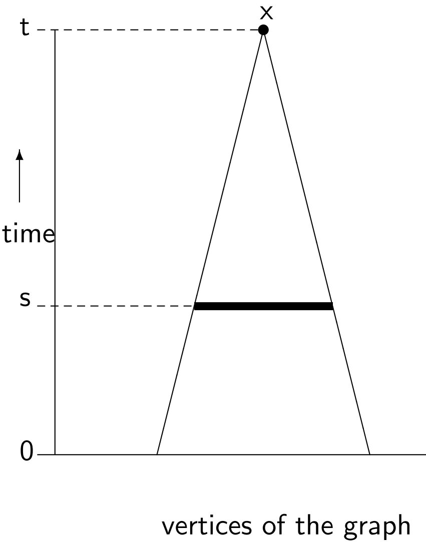


Cone argument for upper bound for $\tau(\epsilon)$

Need to bound the difference $A_t - C_t$.

x is covered at time t by a *center* born at time s if
the *center* lies in the corresponding cross section of the space-time cone $K_{x,t} := \{(y,s) \in \Gamma(N) \times [0,t] : |y - x| \leq (t-s)/\sqrt{2\pi}\}$.



Upper bound for $\tau(\epsilon)$ (continued)

$$\text{So } P(x \notin \mathcal{C}_t | s_0, s_1, s_2, \ldots) = \prod_{i: s_i \le t} \left[1 - \frac{(t - s_i)^2}{2N^2}\right] \le \exp\left[-\sum_{i: s_i \le t} \frac{(t - s_i)^2}{2N^2}\right].$$

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$$\le \exp\left[- \sum_{i: s_i \le t} \frac{(t - s_i)^2}{2N^2} \right].$$

This together with the inequality $1 - e^{-x} \ge x - x^2/2$ gives

$$EC_t \ge N^2 E\left[1 - \exp\left(- \int_0^t \frac{(t-s)^2}{2N^2}\, d\tilde{X}_s \right) \right]$$
$$\ge \frac{t^2}{2} + \int_0^t \frac{(t-s)^2}{2} \lambda EC_s ds - \frac{EA_t^2}{2N^2}.$$

Upper bound for $\tau(\epsilon)$ (continued)

$$\text{So } P(x \notin \mathcal{C}_t | s_0, s_1, s_2, \ldots) = \prod_{i: s_i \le t} \left[1 - \frac{(t - s_i)^2}{2N^2}\right] \le \exp\left[- \sum_{i: s_i \le t} \frac{(t - s_i)^2}{2N^2}\right].$$

This together with the inequality $1 - e^{-x} \ge x - x^2/2$ gives

$$EC_t \ge N^2 E\left[1 - \exp\left(- \int_0^t \frac{(t-s)^2}{2N^2}\, d\tilde{X}_s\right)\right] \ge \frac{t^2}{2} + \int_0^t \frac{(t-s)^2}{2} \lambda EC_s ds - \frac{EA_t^2}{2N^2}.$$

$$\text{Also } EA_t = \frac{t^2}{2} + \int_0^t \frac{(t-s)^2}{2} \lambda EA_s\, ds.$$

Upper bound for $\tau(\epsilon)$ (continued)

$$\text{So } P(x \notin \mathcal{C}_t | s_0, s_1, s_2, \ldots) = \prod_{i:s_i \le t} \left[1 - \frac{(t-s_i)^2}{2N^2}\right] \le \exp\left[-\sum_{i:s_i \le t} \frac{(t-s_i)^2}{2N^2}\right].$$

This together with the inequality $1 - e^{-x} \ge x - x^2/2$ gives

$$EC_t \ge N^2 E\left[1 - \exp\left(-\int_0^t \frac{(t-s)^2}{2N^2}\, d\tilde{X}_s\right)\right] \ge \frac{t^2}{2} + \int_0^t \frac{(t-s)^2}{2} \lambda EC_s ds - \frac{EA_t^2}{2N^2}.$$

$$\text{Also } EA_t = \frac{t^2}{2} + \int_0^t \frac{(t-s)^2}{2} \lambda EA_s\, ds.$$

So $EA_t - EC_t$ satisfies a renewal inequality.

Upper bound for $\tau(\epsilon)$ (continued)

- From the last argument

$$EC_t \geq EA_t - C\frac{a^2(t)}{N^2}.(\text{ recall } EA_t \sim a(t))$$

- Using Markov inequality we can bound $A_t - C_t$, and have

Lemma

For any $\gamma > 0$,
$\limsup_{N\to\infty} P[\tau(\epsilon) > \sigma((1+\gamma)\epsilon)] \leq P\left(M \leq (1+\gamma)\epsilon^{1/3}\right) + C\frac{\epsilon^{1/3}}{\gamma}.$

Remark: So $\tau(\epsilon) \sim (2 - 2\alpha/3)N^{\alpha/3}\log N.$

How does C_t/N^2 grow?

- Choose $\psi(t) := N^{\alpha/3}[(2-2\alpha/3)\log N - \log(M) + t]$ so that

$$N^{-2}A_{\psi(t)} \xrightarrow{P} e^t/3, -\infty < t < \infty.$$

  In particular for $W = \psi(\log(3\epsilon))$, $N^{-2}A_W \xrightarrow{P} \epsilon$.

- If ϵ is small, then the bound on $A_t - C_t$ suggests $C_W \approx (\epsilon - O(\epsilon^2))N^2$ w. h. p.

To study the growth of $\mathcal{C}_t$ after time W,

- call the *centers* present at time W 'generation 0 centers'.
- For $k \geq 1$, generation k *centers* are those which are born from area covered by generation $(k-1)$ *centers*.

Def: For $k \geq 0$ let $C^k_{W,\psi(t)}$ (resp $A^k_{W,\psi(t)}$) be the area covered in $\mathcal{C}_t$ (resp $\mathcal{A}_t$) by *centers* of generations $j \in \{0, 1, \ldots, k\}$.

Estimates for area covered by generation 0 centers

$A^0_{W,\psi(t)}$ can be expressed in terms of X_W, L_W and A_W, and using their limiting behavior if

$$g_0(t) := \epsilon[1 + (t - \log(3\epsilon)) + (t - \log(3\epsilon))^2/2], \text{ then}$$
$$N^{-2} A^0_{W,\psi(s)} \xrightarrow{P} g_0(s) \quad \text{uniformly for } s \in [\log(3\epsilon), t].$$

Using another cone argument we bound $EA^0_{s,t} - EC^0_{s,t}$, which shows that if $\eta > 0$ is small, then

$$\text{w.h.p. } N^{-2}\left(C^0_{W,\psi(s)} - A^0_{W,\psi(s)}\right) \geq -\epsilon^{1+\eta} \ \forall\, s \in [\log(3\epsilon), t].$$

So for small ϵ, $g_0(t)$ and $f_0(t) := g_0(t) - \epsilon^{1+\eta}$ provide upper and lower bounds respectively for $N^{-2} C^0_{W,\psi(t)}$ w.h.p.

Lower bound for $C^1_{W,\psi(t)}$

A point $x \notin \mathcal{C}^1_{W,\psi(t)}$, if $x \notin \mathcal{C}^0_{W,\psi(t)}$ and no generation 1 *center* is born in the space-time cone

$$K^{\epsilon}_{x,t} \equiv \left\{(y,s) \in \Gamma(N) \times [W, \psi(t)] : |y-x| \leq (\psi(t)-s)/\sqrt{2\pi}\right\}.$$

Comparing with an appropriate Poisson process,

$$\text{w.h.p.} \;\; N^{-2} C^1_{W,\psi(s)} \geq f_1(s) - \delta \;\; \forall\, s \in [\log(3\epsilon), t].$$

for any $\delta > 0$ and small ϵ, where

$$1 - f_1(t) = (1 - f_0(t)) \exp\left(-\int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} f_0(s)\, ds\right).$$

Lower bound for $C_{\psi(t)}$

The last argument can be iterated. $\{f_k(\cdot)\}$ satisfying

$$1-f_{k+1}(t) = (1-f_k(t)) \exp\left(-\int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2}(f_k(s) - f_{k-1}(s))\, ds\right)$$

provides a lower bound for $C^k_{W,\psi(\cdot)}$.

$f_k \uparrow f_\epsilon$ uniformly, where f_ϵ satisfies

$$f_\epsilon(t) = 1 - (1 - f_0(t)) \exp\left(-\int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} f_\epsilon(s)\, ds\right)$$

with $f_\epsilon(\log(3\epsilon)) = \epsilon - \epsilon^{1+\eta}$. Choosing k large and ϵ small, for any $\delta > 0$,

$$\text{w.h.p. } N^{-2} C_{\psi(s)} \geq f_\epsilon(s)) - \delta \ \ \forall\, s \in [\log(3\epsilon), t].$$

Upper bound for $C_{\psi(t)}$

Recall that $g_0(\cdot) = \epsilon[1 + (\cdot - \log(3\epsilon)) + (\cdot - \log(3\epsilon))^2/2]$ is an upper bound of $C^0_{W,\psi(t)}$. Following the argument for lower bound and noting that $C^k_{W,\psi(t)} \uparrow C_{\psi(t)}$ uniformly in k, if

$$g_\epsilon(t) = 1 - (1 - g_0(t)) \exp\left(- \int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} g_\epsilon(s)\, ds \right), \text{ then}$$

$$\text{w.h.p. } N^{-2} C_{\psi(s)} \leq g_\epsilon(s) + \delta \;\; \forall\, s \in [\log(3\epsilon), t].$$

Limiting behavior of $C_{\psi(t)}$

$g_\epsilon(t)$ and $f_\epsilon(t)$ provide upper and lower bounds for $C_{\psi(t)}$. In the limit as $\epsilon \to 0$ both the bounds converge to the same thing.
Let $h_\epsilon(t) = e^t/3$ for $t < \log(3\epsilon)$.

$$h_\epsilon(t) = 1 - \exp\left(-\int_{-\infty}^{\log(3\epsilon)} \frac{(t-s)^2}{2}\frac{e^s}{3}\,ds - \int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} h_\epsilon(s)\,ds\right)$$

for $t \geq \log(3\epsilon)$. Then as $\epsilon \to 0$,

$f_\epsilon(s) - h_\epsilon(s)$ and $g_\epsilon(s) - h_\epsilon(s) \to 0$ uniformly in $s \in [\log(3\epsilon), t]$, and

$h_\epsilon(t) \to h(t)$ satisfying (a) $\lim_{t\to-\infty} h(t) = 0$ (b) $\lim_{t\to\infty} h(t) = 1$ (c) h is increasing with $0 < h(t) < 1$ and

$$(d) \quad h(t) = 1 - \exp\left(-\int_{-\infty}^{t} \frac{(t-s)^2}{2} h(s)\,ds\right).$$

Limiting behavior of $C_{\psi(t)}$

The upper and lower bounds for $C_{\psi(\cdot)}$ can be combined to have

$$\lim_{N\to\infty} P\left(\sup_{s\le t} |N^{-2}C_{\psi(s)} - h(s)| \le \delta\right) = 1$$

for any $t < \infty$ and $\delta > 0$

Remarks:

- The displacement of $\tau(\epsilon)$ from $(2 - 2\alpha/3)N^{\alpha/3}\log N$ on the scale $N^{\alpha/3}$ is dictated by the random variable M that gives the rate of growth of the balloon branching process.
- Once C_t reaches ϵN^2, the growth is deterministic.
- There is a cutoff phenomenon as the fraction of covered area reaches a small level in time $O(N^{\alpha/3}\log N)$ and then onwards it increases to 1 within time $O(N^{\alpha/3})$.

The cover time T_N

- $h(t)$ never reaches 1.
- Since $N^{-2} C_{\psi(s)} \sim h(s)$, the number of *centers* in $\mathcal{C}_{\psi(0)}$ dominates a Poisson random variable with mean

$$\lambda(\delta) N^{2-2\alpha/3}, \text{ where } \lambda(\delta) = \int_{-\infty}^{0} (h(s) - \delta)^+ \, ds,$$

which are uniformly distributed in the torus.
- If $\delta > 0$ is small, then $\lambda(\delta) > 0$.
- Divide the torus into smaller squares with side $\kappa N^{\alpha/3} \sqrt{\log N}$.
- With high probability each of the small squares owns at least one *center* at time $\psi(0)$.
- This makes $T_N \leq \psi(0) + O(N^{\alpha/3} \sqrt{\log N})$, and so

$$T_N / N^{\alpha/3} \log N \to 2 - 2\alpha/3.$$

Thank You